# A study on discrete Ponzi Scheme model through Sturm-Liouville theory

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**Abstract:** In this paper, we introduce a second order self-adjoint difference equation which describes the dynamics of Ponzi schemes: a type of investment fraud that promises more than it can deliver. We use the Sturm-Liouville theory to study the discrete equation with boundary conditions. The model is based on a promised, unrealistic interest rate $r_p$, a realised nominal interest rate $r_n$, a growth rate of the deposits $r_i$, and a withdrawal rate $r_w$. Giving some restrictions on the rates $r_p, r_i$, and $r_w$, we prove some theorems to when the fund will collapse or be solvent. Two examples are given to illustrate the applicability of the main results.

**Keywords:** Ponzi scheme; difference equation; Sturm-Liouville boundary value problem; Green's function; discrete calculus; Charles Ponzi; investment; rate of return.

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## 1 Introduction

In late 1919 Boston, Italian immigrant Charles Ponzi conducted a type of investment fraud that would soon become synonymous with his name: a Ponzi scheme. The main characteristic of a Ponzi scheme is the payment of old investors with the money from new investors. To attract individuals into taking part of the scam, a Ponzi schemer will usually offer a rate of return that is above the market rate. However, since this promised rate of return is fictitious, the fund can quickly become depleted as investors begin to make withdrawals, and soon the Ponzi schemer will run out of funds and be caught for their crime. This forces the schemer to recruit an ever-increasing number of individuals into the fraud to postpone a collapse. Realistically however, the pool of new available funds should eventually dry up, which implies that there is an inevitable end to every Ponzi scheme.

In the last 100 years since Charles Ponzi, the number of Ponzi scheme occurrences has grown (Lewis, 2012). While they usually are not memorable news stories (if they even make it into the news at all) the effect they can have on the life of the scammed individuals is very real and often devastating. Many people have lost their life’s savings as a result of Ponzi schemes. While the government often tries to reimburse those who lose money as a result of a Ponzi scheme collapse, it is often unable to provide full compensation to all affected.

Due to the substantial impact that Ponzi schemes can have on the lives of many individuals and economic areas, it is important to understand their economic and mathematical underpinnings. The literature on the sustainability of Ponzi schemes is not large, but there are a few key papers that attempted to understand the mechanics of Ponzi schemes. Some of them deal with how Ponzi schemes can spread and recruit new individuals (Zhu et al., 2017; Bhattacharya, 2003; Carpio, 2011). Others analyse the one-on-one interactions between a single investor and schemer and apply game theory to see when it is most beneficial to join, quit, or expose a fraudulent investment (Tennant, 2011).

In Artzrouni (2009), the author constructed a Ponzi Scheme model in continuous time. The model is a system of three linear first order differential equations of deposit, withdrawal, and total money in the fund functions in time. The model has been considered as an initial value problem (IVP). Then the author analysed the unique solution of the IVP to obtain some conditions on the rates to have the Ponzi model collapse or be solvent (mathematically speaking, it has a zero value in time or only positive values over time). The analysis has been done with the use of some algebra. Motivated by this paper and the others we mentioned above, we aim to improve the existence of Ponzi models by introducing a more realistic model in discrete time. We consider the model as a boundary value problem and use the Sturm-Liouville theory to analyse it.

The plan of the paper is as follows: in Section 2, we give some preliminaries so that the reader will be familiar with the mathematical formulations in the later sections. In Section 3, we introduce the model as a system of linear first order difference equations. We consider the model in two cases: a non-constant withdrawal rate and a constant withdrawal rate. Then we focus on the model with a constant withdrawal rate and solve the system as an IVP. This section ends with a theorem which gives a condition on the rates so that the

Ponzi model stays solvent over time. In Section 4, we continue to work on the model with a constant withdrawal rate. We introduce the boundary value problem (BVP) where the equation is self-adjoint. We then solve the BVP to obtain the main results of this paper. We state and prove a theorem which has conditions on the rates and right end-point to obtain that the Ponzi model collapses over time. We demonstrate the applicability of the theorems by giving two real-life examples in Section 5. In Section 6, we discuss advantages of the discrete model for investors and government regulators to have better and more reliable ways of predictions.

## 2 Preliminaries

Let $a \in \mathbb{R}$, $\mathbb{N}_a = \{a,\ a+1,\ a+2, \ldots\}$ .

The backward difference operator, or nabla operator $(\nabla)$ for a function $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ is defined by

$$(\nabla f)(t) = f(t) - f(t-1)) .$$

The forward difference operator, or delta operator $(\Delta)$ for a function $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ is defined by

$$(\Delta f)(t) = f(t+1) - f(t).$$

We define discrete interval as a set of the form

$$\mathbb{N}_a^b = \{a, a+1, ..., b\}$$

where $a, b \in \mathbb{R}$ and $b - a$ is a positive integer.

**Theorem 2.1 (**Atıcı et al., 2019**):** *Assume* $\lambda \in \mathbb{R} \backslash \{-1\}$. *The first order nabla difference equation*

$$\nabla y(t) = \lambda y(t-1) + f(t-1) \quad for \quad t \in \mathbb{N}_1, \tag{2.1}$$

*has the general solution*

$$y(t) = (1+\lambda)^t c + \sum_{s=0}^{t-1} (1+\lambda)^{t-s-1} f(s), \quad t \in \mathbb{N}_1, \tag{2.2}$$

*where $c$ is constant.*

The following boundary value problem has been extensively studied in the literature (Atıcı and Guseinov, 2002: Aykut and Guseinov, 2003; Anderson et al. 2006):

$$-\Delta[p(t-1)\Delta y(t-1)] + q(t)y(t) = h(t), \quad t \in \mathbb{N}_a^b, \tag{2.3}$$

$$\alpha y(a-1) - \beta y^{[\Delta]}(a-1) = 0, \quad \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \tag{2.4}$$

where $a, b \in \mathbb{Z}$, $\alpha, \beta, \gamma, \delta$ are constants such that $|\alpha| + |\beta| \neq 0$ and $|\gamma| + |\delta| \neq 0$. Here the notation $y^{[\Delta]}(t)$ is used for $p(t)\Delta y(t)$.

Let the functions $\varphi$ and $\psi$ be the solutions of the corresponding homogeneous equation

$$-\Delta[p(t-1)\Delta y(t-1)] + q(t)y(t) = 0, \quad t \in \mathbb{N}_a^b,$$

under the following initial conditions

$$\varphi(a-1) = \beta, \quad \varphi^{[\Delta]}(a-1) = \alpha$$
$$\psi(b) = \delta, \quad \psi^{[\Delta]}(b) = -\gamma.$$

**Theorem 2.2:** *The solution of the nonhomogeneous equation (2.3) with nonhomogeneous boundary conditions*

$$\alpha y(a-1) - \beta y^{[\Delta]}(a-1) = d_1, \quad \gamma y(b) + \delta y^{[\Delta]}(b) = d_2, \tag{2.5}$$

*is given by*

$$y(t) = \frac{d_2}{D}\varphi(t) + \frac{d_1}{D}\psi(t) + \sum_{s=a}^{b} G(t,s)h(s), \tag{2.6}$$

*where $D = -W[\varphi, \psi](t)$ is the negative of the Wronskian and $G(t, s)$ is the Green's function of the associated BVP given by*

$$G(t,s) = \frac{1}{D}\begin{cases} \varphi(t)\psi(s), & a-1 \leq t \leq s \leq b+1 \\ \psi(t)\varphi(s), & a-1 \leq s \leq t \leq b+1. \end{cases}$$

*Proof*: Since the Wronskian is constant, we have

$$D = \varphi^{[\Delta]}(a-1)\psi(a-1) - \varphi(a-1)\psi^{[\Delta]}(a-1) = \alpha\psi(a-1) - \beta\psi^{[\Delta]}(a-1)$$
$$= \varphi^{[\Delta]}(b)\psi(b) - \varphi(b)\psi^{[\Delta]}(b) = \gamma\varphi(b) + \delta\varphi^{[\Delta]}(b).$$

We want to point out that $D$ is nonzero here and its proof can be found in the paper (Atıcı and Guseinov, 2002). The homogeneous solution of the equation is $y_h(t) = C_1\varphi(t) + C_2\psi(t)$, where $C_1$ and $C_2$ are constants. Here we determine $C_1$ and $C_2$ by use of the nonhomogeneous boundary conditions (2.5). Hence we have $C_1 = \dfrac{d_2}{D}$ and $C_2 = \dfrac{d_1}{D}$.
One can easily verify that the particular solution of the nonhomogeneous equation is

$$y_p(t) = \sum_{s=a}^{b} G(t,s)h(s).$$

□

**Remark 2.1:** In several published papers (Atıcı and Guseinov, 2002: Aykut and Guseinov, 2003; Anderson et al. 2006), the formula given in equation (2.6) mistakenly expressed as

$$y(t) = \frac{d_2}{D}\varphi(t) - \frac{d_1}{D}\psi(t) + \sum_{s=a}^{b} G(t,s)h(s).$$

Since it was given as a remark or a note in the papers, we stated and proved the theorem for reader's convenience.

For further reading on discrete calculus, we refer the reader to a book by Kelley and Peterson, (2001).

## 3 Discrete Ponzi scheme model

The model has the following notations and assumptions:

$r_p$ the promised, unrealistic interest rate

$r_n$ the realised nominal interest rate

$r_i$ the growth rate of the deposits

$r_{w_t}$ the withdrawal rate in time

$r_w$ constant withdrawal rate

$S_t$ the amount of money the schemer possesses

$D_t$ the cash inflow (deposit) function

$W_t$ the cash outflow (withdrawal) function

A1. $0 < r_p, r_i$ and $0 \leq r_n$

A2. $0 < r_w < 1$

A3. $(1 + r_p)(1 - r_w) \neq (1 + r_n)$.

We start by defining the functions that characterise the evolution of the fund controlled by a Ponzi schemer. Let $S_t$ be the amount of money in the investment fund at the start of time $t$, $W_t$ be the amount of money withdrawn from the fund at time $t$, and $D_t$ be the amount of money deposited into the fund at time $t$. These functions are related together through the following recurrence relation:

$$S_{t+1} = (1 + r_n)S_t + D_t - W_t,$$

where $r_n$ is the market rate of return. The sequence of events progresses as follows: in each time period, the first thing the manager does is observe how much money is currently in their fund, $S_t$. They invest this amount in the market and earn back their principal plus interest, $(1 + r_n)S_t$. Next, an amount $D_t$ is added, representing the money being deposited by new (and potentially some old) investors, and an amount $W_t$ is removed, representing the money being taken out of the fund by investors that wish to leave. Finally, at the end of the time period, this amount $(1 + r_n)S_t + D_t - W_t$ is observed by the manager, and becomes $S_{t+1}$ for the next time period.

We now choose a functional form for both $D_t$ and $W_t$. First, consider $D_t$. A simple, workable condition is to require that deposits increase at a constant exponential rate. While in reality the growth may be stochastic, we assume a deterministic setting for our analysis; further research involving probabilities would enter into the field of ruin theory, and is a possibility for the future. In a discrete setting, this means that deposits will follow $D_{t+1} =$

$(1+r_i)D_t$, where $r_i$ is the exogenous growth rate of deposits. Solving this equation results in the function

$$D_t = D_0(1+r_i)^t,$$

where $D_0 > 0$ is the exogenous initial amount of money that investors give to the Ponzi schemer. This deposit function is an aggregation of money, and does not attempt to explain the specifics of its composition. It may be that there are no new investors and the current investors decide to invest more/again. It could also be that the previous investors decide not to invest again, and the new money is coming entirely from new investors. Any intermediary between these extremes is permissible, and this flexibility applies to withdrawals too.

For $W_t$ we assume that in each time period the amount of money that leaves the fund is some percentage of the cumulative amount of money that has been deposited and not already withdrawn. This percentage, $r_w$ is exogenously given. Furthermore, this principal amount returned to investors is multiplied by $(1+r_p)$, where $r_p$ is the rate of return that the manager promises investors and is another choice variable for the fund manager. $r_p$ is decided upon by the manager at the outset of the operation, and is constant over $\mathbb{N}_0$. The function is constructed as follows:

$$\begin{aligned}
&W_0 = 0\\
&W_1 = r_{w_1}(1+r_p)D_0\\
&W_2 = r_{w_2}((1+r_p)D_1 + (1-r_{w_1})(1+r_p)^2 D_0)\\
&W_3 = r_{w_3}((1+r_p)D_2 + (1-r_{w_2})(1+r_p)^2 D_1 + (1-r_{w_2})(1+r_{w_1})(1+r_p)^3 D_0)
\end{aligned}$$

with the general equation being

$$W_t = r_{w_t} \sum_{k=0}^{t-1} \prod_{i=1}^{t-k-1} (1-r_{w_i})(1+r_p)^{t-k}(1+r_i)^k D_0$$

with

$$W_t = \sum_{k=0}^{t-1} (1+r_p)^{t-k} r_w (1-r_w)^{t-1-k} D_0 (1+r_i)^k.$$

as the special case when the withdrawal rate is constant.

These functions show that for any deposit $D_t$, the amount of money withdrawn due to that deposit principal in $j$ time units will be $r_{w_{t+j}} \prod_{i=t+1}^{t+j-1}(1-r_{w_i})(1+r_p)^j D_t$. This reflects two things: one, the longer money is kept in the fund, the more interest the deposit accrues which the manager will have to pay out, and two, the amount of principal in the fund decays over time due to withdrawals.

We now consider the following system of equations

$$\begin{cases}
W_t &= r_{w_t}(1+r_p)D_{t-1} + r_{w_t}(1+r_p)\dfrac{(1-r_{w_{t-1}})}{r_{w_{t-1}}}W_{t-1}\\
S_t &= (1+r_n)S_{t-1} + D_{t-1} - W_{t-1}\\
D_t &= (1+r_i)D_{t-1}.
\end{cases} \tag{3.1}$$

The nabla-difference equation of the withdrawal function is

$$\nabla W_t = r_{w_t}(1+r_p)D_{t-1} + (r_{w_t}(1+r_p)\frac{(1-r_{w_{t-1}})}{r_{w_{t-1}}} - 1)W_{t-1}.$$

The nabla-difference equation of the fund is

$$\nabla S_t = r_n S_{t-1} + D_{t-1} - W_{t-1}.$$

Taking the delta-derivative of the fund's equation results in

$$\Delta\nabla S_t - (r_n + \xi_t + \xi_t r_n)\nabla S_t + \xi_t r_n S_t = \nabla D_t + [1 - \frac{r_{w_t}}{r_{w_{t-1}}}(1+r_p)]D_{t-1},$$

where $\xi_t = r_{w_t}(1+r_p)\frac{(1-r_{w_{t-1}})}{r_{w_{t-1}}} - 1$.

To put this in a self-adjoint form, we consider $\Pi_{s=1}^{t}(1+m_s)$, where $m_t = r_n + \xi_t + \xi_t r_n$. Dividing both sides by this expression results in the self-adjoint form of

$$\begin{aligned} &-\Delta\left[\frac{1}{\Pi_{s=1}^{t-1}(1+m_s)}\Delta S_{t-1}\right] + \frac{1}{\Pi_{s=1}^{t}(1+m_s)}(-\xi_t r_n)S_t \\ &\quad = \frac{-1}{\Pi_{s=1}^{t}(1+m_s)}(\nabla D_t + [1 - \frac{r_{w_t}}{r_{w_{t-1}}}(1+r_p)]D_{t-1}). \end{aligned}$$

Next we solve the IVP $D_t = (1+r_i)D_{t-1}$ with initial condition $D_0 > 0$. Hence we have $D_t = D_0(1+r_i)^t$. We replace this solution in the self-adjoint equation above to have the final form of the equation

$$\begin{aligned} &-\Delta[\frac{1}{\Pi_{s=1}^{t-1}(1+m_s)}\Delta S_{t-1}] + \frac{1}{\Pi_{s=1}^{t}(1+m_s)}(-\xi_t r_n)S_t \\ &\quad = D_0\frac{(1+r_i)^{t-1}([1 - \frac{r_{w_t}}{r_{w_{t-1}}}(1+r_p)] - r_i)}{\Pi_{s=1}^{t}(1+m_s)}. \end{aligned} \tag{3.2}$$

## 3.1 *Constant withdrawal rate*

If we consider $r_{w_t}$ = constant, then the self-adjoint equation (3.2) becomes

$$\begin{aligned} &-\Delta[(1+m)^{-(t-1)}\Delta S_{t-1}] + (1+m)^{-t}(-\xi r_n)S_t \\ &\quad = D_0(r_p - r_i)(1+m)^{-t}(1+r_i)^{t-1}, \end{aligned} \tag{3.3}$$

where $m = r_n + \xi + \xi r_n$ and $\xi = (1-r_w)(1+r_p) - 1$.

Next we consider the system (3.1) with the initial conditions $S_0 \geq 0, D_0 > 0$ and $W_0 = 0$

$$\begin{cases} W_t &= r_w(1+r_p)D_{t-1} + (1+r_p)(1-r_w)W_{t-1} \\ S_t &= (1+r_n)S_{t-1} + D_{t-1} - W_{t-1} \\ D_t &= (1+r_i)D_{t-1}. \end{cases}$$

We start solving this system for $D_t$ in the third equation and we have $D_t = D_0(1+r_i)^t$. Plugging this solution in the first and second equations gives the following system

$$\begin{cases} W_t &= r_w(1+r_p)D_0(1+r_i)^{t-1} + (1+r_p)(1-r_w)W_{t-1} \\ S_t &= (1+r_n)S_{t-1} + D_0(1+r_i)^{t-1} - W_{t-1}. \end{cases}$$

We want to point out that this system provides us a sign to produce a second order difference equation. This is why we study second order difference equations with boundary conditions in the next section. If we continue solving the above IVP, we obtain a solution for $W_t$ as

$$W(t) = D_0(1+r_p)r_w \frac{(1+r_p)^t(1-r_w)^t - (1+r_i)^t}{(1+r_p)(1-r_w) - (1+r_i)}.$$

Plugging this solution in the second equation gives the following first order difference equation:

$$\begin{aligned} S_t = (1+r_n)S_{t-1} + D_0(1+r_i)^{t-1} \\ -D_0(1+r_p)r_w \frac{(1+r_p)^{t-1}(1-r_w)^{t-1} - (1+r_i)^{t-1}}{(1+r_p)(1-r_w) - (1+r_i)}. \end{aligned}$$

Next, we use Theorem 2.1 as a tool to obtain the unique solution. Hence we have

$$S_t = (1+r_n)^t S_0 + \sum_{s=0}^{t-1} (1+r_n)^{t-s-1}(D_s - W_s), \quad t \in \mathbb{N}_0. \tag{3.4}$$

**Theorem 3.1:** *If $r_p < r_i$, then the solution (3.4) of the IVP is positive on $\mathbb{N}_1$.*

*Proof*: We claim that $S_t$ is positive on $\mathbb{N}_1$ if $D_t > W_t$ for all $t \in \mathbb{N}_1$. Hence we show that $D_t - W_t > 0$ for all $t \in \mathbb{N}_1$. Then we have

$$\begin{aligned} D_t - W_t &= D_0(1+r_i)^t - D_0(1+r_p)r_w \frac{(1+r_p)^t(1-r_w)^t - (1+r_i)^t}{(1+r_p)(1-r_w) - (1+r_i)} \\ &= D_0(1+r_i)^t \left[ 1 - (1+r_p)r_w \frac{\dfrac{(1+r_p)^t(1-r_w)^t}{(1+r_i)^t} - 1}{(1+r_p)(1-r_w) - (1+r_i)} \right] \\ &> 0 \end{aligned}$$

for all $t \in \mathbb{N}_1$, if and only if

$$\frac{1 - \left[\dfrac{(1+r_p)(1-r_w)}{1+r_i}\right]^t}{\dfrac{(1+r_i)}{r_w(1+r_p)} - \dfrac{(1-r_w)}{r_w}} < 1$$

for all $t \in \mathbb{N}_1$.

The numerator of the last quantity is less than 1 since $(1+r_p)(1-r_w) < 1+r_i$. The denominator is greater than 1 since $r_i > r_p$. This completes the proof. $\square$

## 4 Main results

Let $b$ be a positive integer such that $b > 1$. We set the boundary conditions at 0 and $b$ as $S_0$ and $S_b$ respectively, both of which are non-negative real numbers.

Next we have a closer look at the discrete equation $(3.3)$ with boundary conditions at 0 and $b$ on the discrete interval $\mathbb{N}_1^b$. We use the Theorem 2.2 to write the solution of the BVP in terms of its associated Green's function. Hence we have

$$S_t = \frac{S_b}{D}\varphi_t + \frac{S_0}{D}\psi_t + \sum_{s=1}^{b} G(t,s)H(s), \tag{4.1}$$

for $t \in \mathbb{N}_1^{b+1}$, where $H(t) = D_0(r_p - r_i)(1+m)^{-t}(1+r_i)^{t-1}$.

The Green's function for the associated BVP is

$$G(t,s) = \frac{1}{D}\begin{cases} \varphi_t\psi_s, & 0 \le t \le s \le b+1 \\ \psi_t\varphi_s, & 0 \le s \le t \le b+1, \end{cases}$$

where

$$\varphi_t = \frac{(1+r_p)^t(1-r_w)^t - (1+r_n)^t}{(1+r_p)(1-r_w) - (1+r_n)}$$

$$\psi_t = \frac{(1+r_p)^b(1-r_w)^b(1+r_n)^t - (1+r_n)^b(1+r_p)^t(1-r_w)^t}{(1+r_p)(1-r_w) - (1+r_n)}$$

$$D = \frac{(1+r_p)^b(1-r_w)^b - (1+r_n)^b}{(1+r_p)(1-r_w) - (1+r_n)}.$$

We note that the functions $\varphi$ and $\psi$ satisfy the following initial conditions

$$\varphi_0 = 0, \quad \varphi_1 = 1$$
$$\psi_b = 0, \quad \psi_{b+1} = -(1+m)^b.$$

**Theorem 4.1:** *The functions $\varphi$ and $\psi$ possess the following properties:*

- $\varphi_t > 0, \quad t \in \mathbb{N}_1^{b+1}$
- $\psi_t > 0, \quad t \in \mathbb{N}_0^{b-1}$
- $D > 0$
- $\Delta\varphi_t \ge 0, \quad t \in \mathbb{N}_0^b$
- $(\Delta\psi_t)(b-1) < 0.$

*Proof*: The proofs of $i) - iii)$ can be done considering two cases: $(1+r_p)(1-r_w) < (1+r_n)$ and $(1+r_p)(1-r_w) > (1+r_n)$.

*Case 1*. If $(1+r_p)(1-r_w) < (1+r_n)$, then $(1+r_p)^t(1-r_w)^t < (1+r_n)^t$ for all $\in \mathbb{N}_1$. This implies that $\varphi_t$ and $D$ are positive. To prove $ii)$, we have

$$\begin{aligned}\psi_t &= \frac{(1+r_p)^b(1-r_w)^b(1+r_n)^t - (1+r_n)^b(1+r_p)^t(1-r_w)^t}{(1+r_p)(1-r_w)-(1+r_n)} \\ &= (1+r_p)^b(1-r_w)^b(1+r_n)^t\left[\frac{1-(1+r_n)^{b-t}(1+r_p)^{t-b}(1-r_w)^{t-b}}{(1+r_p)(1-r_w)-(1+r_n)}\right] \\ &> 0\end{aligned}$$

for $t \in \mathbb{N}_0^{b-1}$ since $(1+r_p)^{b-t}(1-r_w)^{b-t} < (1+r_n)^{b-t}$ for all $\in \mathbb{N}_0^{b-1}$.

The proof of Case 2 is similar.

The proof of $iv)$ can be done considering two cases: $(1+r_p)(1-r_w) < 1$ and $(1+r_p)(1-r_w) > 1$. In the second case, we have two subcases: $(1+r_p)(1-r_w) < (1+r_n)$ and $(1+r_p)(1-r_w) > (1+r_n)$.

*Case 1*: If $(1+r_p)(1-r_w) < 1$, then $(1+r_p)(1-r_w) < 1+r_n$. Hence we have

$$\Delta\varphi_t = \frac{\left[(1+r_p)(1-r_w)-1\right](1+r_p)^t(1-r_w)^t - r_n(1+r_n)^t}{(1+r_p)(1-r_w)-(1+r_n)} > 0$$

for $t \in \mathbb{N}_0$.

*Case 2*: Let $(1+r_p)(1-r_w) > 1$ and $(1+r_p)(1-r_w) < 1+r_n$. Then we have

$$\begin{aligned}\Delta\varphi_t &= \frac{\left[(1+r_p)(1-r_w)-1\right](1+r_p)^t(1-r_w)^t - r_n(1+r_n)^t}{(1+r_p)(1-r_w)-(1+r_n)} \\ &= \left[(1+r_p)(1-r_w)-1\right](1+r_p)^t(1-r_w)^t \\ &\qquad \left[\frac{1-\left(\frac{r_n}{(1+r_p)(1-r_w)-1}\right)\left(\frac{(1+r_n)^t}{(1+r_p)^t(1-r_w)^t}\right)}{(1+r_p)(1-r_w)-(1+r_n)}\right] \\ &> 0\end{aligned}$$

for $t \in \mathbb{N}_0$. since $\left(\frac{r_n}{(1+r_p)(1-r_w)-1}\right) < 1$ and $\left(\frac{(1+r_n)^t}{(1+r_p)^t(1-r_w)^t}\right) < 1$. The subcase $(1+r_p)(1-r_w) > 1+r_n$ can be handled similarly.

The proof of $v)$ follows from $ii)$ and the initial condition that the function $\psi$ satisfies. Indeed, we have

$$\begin{aligned}\Delta\psi_t)(b-1) &= \psi(b)-\psi(b-1) \\ &= -\psi(b-1) < 0.\end{aligned}$$

□

**Corallary 4.1:** *The Green's function $G(t,s)$ is nonnegative on $\mathbb{N}_0^b \times \mathbb{N}_0^b$.*

**Corallary 4.2:** *If $r_i < r_p$, then the solution $S_t$ of the discrete equation $(3.3)$ along with the nonnegative boundary conditions at $t = 0$ and $t = b$ is nonnegative on $\mathbb{N}_0^b$.*

**Lemma 4.1:** *Suppose $(1 + r_p)(1 - r_w) > 1$ and $r_i < r_p$. If $S_t$ in equation (4.1) satisfies*

$$S_b < \frac{\psi_{b-1}}{(\Delta\varphi_t)(b-1)}[S_0 + \sum_{s=1}^{b-1} \varphi_s H(s)], \tag{4.2}$$

*then $S_t$ in equation (4.1) satisfies the inequalities $(\Delta S_t)(b-1) < 0$ and $(\Delta^2 S_t)(b-1) < 0$.*

*Proof*: We first prove that $(\Delta^2 S_t)(b-1) < 0$. Indeed, we have that $S_t$ satisfies the following equation

$$\Delta^2 S_{t-1} - ((1 + r_n)(1 + r_p)(1 - r_w) - 1)\Delta S_{t-1} \\ +((1 + r_p)(1 - r_w) - 1)r_n S_t = D_0(r_i - r_p)(1 + r_i)^{t-1}.$$

If we solve this equation for $\Delta^2 S_{t-1}$ and replace $t$ by $b$, we have

$$(\Delta^2 S_t)(b-1) = ((1 + r_n)(1 + r_p)(1 - r_w) - 1)(\Delta S_t)(b-1) \\ -((1 + r_p)(1 - r_w) - 1)r_n S_b + D_0(r_i - r_p)(1 + r_i)^{b-1}.$$

Applying the $\Delta$ operator in equation (4.1) and then replacing $t$ by $b-1$, we have

$$(\Delta S_t)(b-1) = \frac{S_b}{D}(\Delta\varphi_t)(b-1) - \frac{S_0}{D}\psi_{b-1} - \frac{\psi_{b-1}}{D}\sum_{s=1}^{b-1} \varphi_s H(s). \tag{4.3}$$

The inequality in equation (4.2) implies that $(\Delta S_t)(b-1) < 0$. Hence we have the desired result. □

**Theorem 4.2:** *Suppose $(1 + r_p)(1 - r_w) > 1$ and $r_i < r_p$. If there exists $t_0$ such that the unique solution, $S_t$ in equation (3.4), of the IVP with the initial condition $S_0$ satisfies (4.2) for all $t \in \mathbb{N}_{t_0+1}$, then the fund $S_t$ in equation (3.4) is collapsed over time.*

*Proof*: Let $c \in \mathbb{N}_{t_0+1}$ such that the unique solution $S_t$ in equation (3.4), of the IVP with the initial condition $S_0$ satisfies

$$S_c < \frac{\psi_{c-1}}{(\Delta\varphi_t)(c-1)}[S_0 + \sum_{s=1}^{c-1} \varphi_s H(s)].$$

Here we have two cases to consider:

- if $S_c$ is not positive, then there is nothing to prove.
- if $S_c$ is positive, then we consider $S_0$ and $S_c$ as boundary conditions.

In addition, $S_t$ solves the equation in $(3.3)$. Then by Lemma 4.2, $S_t$ is decreasing and concave down at $c-1$. This behaviour of the solution at $c-1$ indicates that the fund is going down. In an iterative way, the solution $S_t$ can be analysed for $t > c$ as we did for the point $c$. The graph of $S_t$ will be decreasing and concave down on $\mathbb{N}_{t_0+1}$. It follows that the fund $S_t$ will be collapsed over time. □

## 5 Illustrative examples

In this section, we consider some realistic rates by analysing Ponzi schemes that have happened in real-life and estimating the model parameters.

**Example 5.1:** On the heels of a Ponzi scheme that cheated investors out of 102 million dollars, in 2018 the Securities and Exchange Commission (SEC) charged a former insurance broker with defrauding inexperienced retail investors (https://www.sec.gov/litigation/litreleases/2018/lr24173.htm). By crudely estimating the rates of this fraud from available information, we have the following construction of the problem.

Estimated rates: $S_0 = 0$, $r_n = 0$, $d_0 = 51475.5$, $r_i = 0.0369027$, $r_p = 0.0466351$, $r_w = 0.0207987$.

We first note that the market rate $r_n$ is not actually $0$, but because the case information seems to indicate that the schemer did not invest any of their funds into any investment opportunities at all, we can set $r_n = 0$ to reflect this. Second, note that $(1+r_p)(1-r_w) > 1$ and $r_i < r_p$. In this problem the time unit is a quarter of a year. Let us define

$$L_b := \frac{\psi_{b-1}}{(\Delta\varphi_t)(b-1)} \sum_{s=1}^{b-1} \varphi_s H(s).$$

Then the solution $S_t$ of the IVP in equation (3.4) satisfies the following:

$$\begin{cases} S_b > L_b, & b \leq 66 \\ S_b < L_b, & b > 66. \end{cases}$$

Theorem 4.3 implies that $S_t$ in equation (3.4) will eventually become zero. Indeed, a collapse occurs at $92 < t < 93$.

**Example 5.2:** In 2018, the Securities and Exchange Commission (SEC) charged a San Diego company, its president, and his business partner with running a multimillion dollar Ponzi scheme that defrauded hundreds of individual investors (https://www.sec.gov/litigation/litreleases/2018/lr24293.htm).

Estimated rates: $S_0 = 0, r_n = 0, d_0 = 42259.6, r_i = 0.108865, r_p = 0.1, r_w = 0.127074.$

We note that $(1+r_p)(1-r_w) < 1+r_i$ and $r_p < r_i$. In this problem the time unit is a month. By Theorem 3.1, we conclude that if the SEC had not shut it down, this Ponzi scheme would have remained solvent over time.

## 6 Conclusion

There exists an incorrect notion about the short, or at least finite, longevity of Ponzi schemes. Most people assume that they must fail. However, we have shown this is not the case. In this paper, we constructed a model that shows how a Ponzi scheme can last theoretically forever when the deposit growth, withdrawal rate, promised rate of return, and market rate of return obey certain inequalities. Granted, in the real-world there exists shocks that could cause the system to collapse, but these inequalities still give us a good idea of how a Ponzi scheme can last a very long time. This question of longetivity, while important when discussing investment frauds such as Ponzi schemes, is also extremely important to discuss when analysing governmental pension plans, such as Social Security.

Writing the model in discrete time makes the model more realistic in the sense that money inflow and outflows cannot be instantaneous. There is always some amount of time between new investors joining and old ones leaving, and when information is given to economic agents, it is often done so in a discrete manner (for example, monthly investment reports or quarterly GDP calculations). It is not intuitive to think of the process's mechanisms as instantaneous. For example, Artzrouni's continuous model's estimation of $r_i = 7.187$ (for the case of Charles Ponzi), which can be called "the instantaneous rate of increase in new investments/deposits", lacks essence because investments do not happen instantaneously. It is only by integrating the deposit function that we reach an interpretation. In contrast, our discrete model's estimate of $r_i = 0.02$ is more descriptive because not only does it have meaning from summing the deposit function, it also tells us how new investments will increase from one defined time period to the next; there is no ambiguity with the meaning of "instantaneous".

Similarly, these critiques apply to the withdrawal function. Besides the points made above, there is an additional comment to make, specifically about the rate of withdrawal, $r_w$. In the continuous model, $r_w$ is simply a parameter that controls the magnitude of withdrawals; the greater $r_w$ is, the more money is 'instantaneously' removed from the fund. However, in our discrete model, $r_w$ receives another property; it is a percentage. It represents the proportion of people/money that leaves each time period. One cannot get this interpretation from certain continuous models, as evidenced by Artzrouni's estimate (again regarding Charles Ponzi) of $r_w = 1.47$.

While not perfect, a discrete model is an attempt to account for a lack of realism in a continuous account of investment operations. With further research, the continuous and discrete models can be unified into model that can work for arbitrary time scales. For the definition and the theory for time scales, we refer the reader to a book (Bohner and Peterson, 2001). Hopefully, expanding our understanding of Ponzi scheme dynamics can lead to better prevention and correction of these types of crimes.

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